Besov Estimates for Sub-elliptic Equations in the Heisenberg Group

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Abstract

In this paper, we deal with weak solutions to non-degenerate sub-elliptic equations in the Heisenberg group, and study the regularities of solutions. We establish horizontal Calderón-Zygmund type estimate in Besov spaces with more general assumptions on coefficients for both homogeneous equations and non-homogeneous equations. This study of regularity estimates expands the Calderón-Zygmund theory in the Heisenberg group.

Keywords: Heisenberg group; Sub-elliptic equations; Regularity; Besov spaces.

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1 Introduction

The main purpose of this article is to study Besov regularities of weak solutions to a class of sub-elliptic equations of the type

$$\operatorname{div}_H A(x, \mathfrak{X}u) = 0 \tag{1.1}$$

and

$$\operatorname{div}_H A\left(x, \mathfrak{X}u\right) = \operatorname{div}_H\left(|F|^{p-2}\, F\right) \tag{1.2}$$

in Ω, where Ω is an open and bounded sub-domain in the Heisenberg group $\mathbb{H}^n = \mathbb{R}^{2n+1}$ ($n \geq 1$). We call (1.1) and (1.2) the homogeneous equation and the non-homogeneous equation, respectively. The unknown $u \in HW^{1,p}_{\text{loc}}(\Omega)$, where the sub-elliptic Sobolev space $HW^{1,p}(\Omega)$ will be introduced in Section 2. In both equations, the horizontal divergence operator div_H and the horizontal gradient $\mathfrak{X}$ are defined by

$$\begin{aligned} \operatorname{div}_H F &= \sum_{i=1}^{2n} X_i F_i, \\ \mathfrak{X}u &= (X_1 u, X_2 u, \ldots, X_{2n-1}u, X_{2n}u) \end{aligned}$$

in the distributional sense. Moreover, $A : \Omega \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is assumed to be a Carathéodory vector field with general growth and uniformly elliptic conditions, that is, there exist constants ν, L, $k > 0$ and $0 < \mu < 1$ such that

$$\text{(A1)} \qquad [A(x,\xi) - A(x,\eta)] \cdot (\xi - \eta) \geq \nu \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2 \,,$$

$$\text{(A2)} \qquad |A(x,\xi) - A(x,\eta)| \leq L \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta| \,,$$

$$\text{(A3)} \qquad |A(x,\xi)| \leq k \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}$$

for every ξ, $\eta \in \mathbb{R}^{2n}$ and for almost all $x \in \Omega$. In (1.2), $F : \Omega \to \mathbb{R}^{2n}$.

The regularity of solutions to elliptic equations in Euclidean spaces $\mathbb{R}^n$ has been well studied by Iwaniec [10], DiBenedetto and Manfredi [7]. Then this theory is extended to the case of general elliptic problems, see in relevant papers [11, 12, 4, 3]. For the nonlinear Calderón-Zygmund estimate in the Heisenberg group, Goldstein and Zatorska-Goldstein [8] deal with the quadratic case $p = 2$. Later on the $HW^{1,p}$ estimates for sub-elliptic equations on $\mathbb{H}^n$ are proved by Mingione, Zatorska-Goldstein and Zhong [14]. They consider the equation of the form

$$\operatorname{div}_H [b(x) a(\mathfrak{X}u)] = \operatorname{div}_H\left(|F|^{p-2} F\right)$$

with $b \in \text{VMO}_{\text{loc}}(\Omega)$.

At present, the studies are concerned with the regularity estimates of weak solutions in Besov spaces in both $\mathbb{R}^n$ and $\mathbb{H}^n$ ([2, 6, 9]). Besov spaces consist of a wide class of

functions compared with the classical Sobolev spaces. Baisón [1] deal with nonlinear elliptic equations in divergence form, and obtain a regularity estimate of weak solutions in Besov spaces. Clop [5] and Lyaghfouri [13] extended the result in Besov spaces by establishing a higher integrability of weak solutions.

For the homogeneous case (1.1), we assume that there exists a function $g \in L^{\frac{Q}{\alpha}}(\Omega)$ $(0 < \alpha < 1)$ such that

$$\text{(A4)} \quad |A(x,\xi) - A(y,\xi)| \leq \operatorname{dist}_{\mathrm{CC}}(x,y)^{\alpha} \left(g(x) + g(y)\right) \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}$$

for almost every $x, y \in \Omega$ and all $\xi \in \mathbb{R}^{2n}$. Here $\operatorname{dist}_{\mathrm{CC}}(x,y)$ is the CC-distance between two points x and y in $\mathbb{H}^n$.

While for the non-homogeneous situation (1.2), we assume that there exists a sequence of measurable non-negative functions $g_k \in L^{\frac{Q}{\alpha}}(\Omega)$ $(k \in \mathbb{N},\ 0 < \alpha < 1)$ satisfying that

$$\text{(A5)} \quad \begin{cases} \displaystyle\sum_{k=1}^{\infty} \|g_k\|^q_{L^{\frac{Q}{\alpha}}(\Omega)} < \infty \quad (1 \leq q < \infty) \\ |A(x,\xi) - A(y,\xi)| \leq \operatorname{dist}_{\mathrm{CC}}(x,y)^{\alpha} \left(g_k(x) + g_k(y)\right) \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}} \end{cases}$$

for $\xi \in \mathbb{R}^{2n}$ and almost all $x, y \in \Omega$ such that $2^{-k} \leq \operatorname{dist}_{\mathrm{CC}}(x,y) < 2^{-k+1}$. According to (A5), we write $\{g_k\}_k \in l^q(L^{\frac{Q}{\alpha}}(\Omega))$ in short.

By introducing an auxiliary function

$$V(\xi) = \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{4}} \xi \tag{1.3}$$

with $\xi \in \mathbb{R}^{2n}$, we present the main results of this article.

Theorem 1.1. *Let $0 < \alpha < 1$ and $2 \leq p < 4$. Assume that A satisfies hypotheses* (A1)-(A4) *with $0 < \mu < 1$. If $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution to* (1.1)*, then $V(\mathfrak{X}u) \in B^{\alpha}_{2,\infty}(\Omega)$ locally.*

Theorem 1.2. *Let $0 < \alpha < 1$, $2 \leq p < 4$, and $1 \leq q < \frac{2Q}{Q-2\alpha}$. Assume that the hypotheses* (A1)-(A3) *and* (A5) *hold. If $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution to* (1.2) *with $0 < \mu < 1$ and $|F|^{p-2}F \in B^{\alpha}_{2,q}(\Omega)$, then $V(\mathfrak{X}u) \in B^{\alpha}_{2,q}(\Omega)$ locally.*

See Section 2 for the definitions of $HW^{1,p}(\Omega)$ and $B^{\alpha}_{2,q}(\Omega)$.

The contribution of the main results is to study a wide class of sub-elliptic equations in the Heisenberg group. Our aim is to obtain a Besov regularity estimate of weak solutions. The hypotheses (A1)-(A4) (or (A5)) shall be an extension of the VMO conditions.

This article is organized as follows. In section 2 we give some definitions and tools such as classical inequalities, and we present two Lemmas relating to the reverse Hölder type inequalities of weak solutions. In section 3 and section 4, we present the proofs of Theorem 1.1 and Theorem 1.2, respectively.

2 Preliminary

2.1 Heisenberg Group

In this section, we collect some basic notations and preliminaries for the Heisenberg group. We denote by $(x,t) = (x_1, x_2, \ldots, x_{2n}, t)$ the coordinates of points of the Heisenberg group $\mathbb{H}^n$. The group structure on $\mathbb{H}^n$ is given by

$$\begin{aligned} & (x_1, x_2, \ldots, x_{2n}, t) \circ (y_1, y_2, \ldots, y_{2n}, s) \\ = & \left(x_1 + y_1, x_2 + y_2, \ldots, x_{2n} + y_{2n}, t + s + \frac{1}{2} \sum_{j=1}^{n} (x_j y_{n+j} - x_{n+j} y_j) \right). \end{aligned}$$

An anisotropic dilation induces a homogeneous norm (gauge) of (x,t) by $\left(|x|^2 + t\right)^{\frac{1}{2}}$. For $j = 1, \ldots, n$, we set

$$X_j = \frac{\partial}{\partial x_j} - \frac{x_{n+j}}{2} \frac{\partial}{\partial t}, \; X_{n+j} = \frac{\partial}{\partial x_{n+j}} + \frac{x_j}{2} \frac{\partial}{\partial t}, \; T = \frac{\partial}{\partial t},$$

which represent a basis of the space of left-invariant vector fields on $\mathbb{H}^n$. The vector field $X_1, X_2, \ldots, X_{2n}$ are called the horizontal vector fields. Then the length of the horizontal gradient is given by

$$|\mathfrak{X} u|^2 = \sum_{j=1}^{2n} (X_j f)^2 .$$

2.2 CC-distance and CC-Balls

By considering the well-known Carnot-Carathéodory metric with CC-distance $\mathrm{dist}_{\mathrm{CC}}$, we define CC-balls by

$$B_R(x_0) = \{ y \in \mathbb{H}^n \mid \mathrm{dist}_{\mathrm{CC}}(x_0, y) < R \}$$

with the center x_0 and radius R. By introducing the homogeneous dimension $Q = 2n+2$, one gets the Lebesgue measure of a CC-ball $|B_R(x_0)| \approx R^Q$.

2.3 Horizontal Sobolev Spaces and Besov Spaces

Let $L^p(\mathbb{H}^n)$ be the Lebesgue space in the Heisenberg group, then the dual space of $L^p(\mathbb{H}^n)$ is $L^{p'}(\mathbb{H}^n)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. The horizontal Sobolev space with its norm is defined by

$$\begin{aligned} HW^{1,p}(\Omega) &= \left\{ u \in L^p(\Omega) \;\middle|\; \mathfrak{X} u \in L^p(\Omega) \right\}, \\ \|u\|_{HW^{1,p}(\Omega)} &= \|u\|_{L^p(\Omega)} + \|\mathfrak{X} u\|_{L^p(\Omega)}. \end{aligned}$$

It is clear that a function $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$, if $u \in HW^{1,p}(\Omega_0)$ for every $\Omega_0 \Subset \Omega$.

Let the parameters $0 < \alpha < 1$, $1 \le p < \infty$, $1 \le q \le \infty$. The Besov spaces $B^{\alpha}_{p,q}(\Omega)$ $(\Omega \subset \mathbb{H}^n)$ with its norm are defined via ([16])

$$\begin{aligned}
\|u\|_{B^{\alpha}_{p,q}(\Omega)} &= \|u\|_{L^p(\Omega)} + [u]_{B^{\alpha}_{p,q}(\Omega)} < \infty, \\
[u]_{B^{\alpha}_{p,q}(\Omega)} &= \begin{cases} \left(\int_{\Omega}\left(\int_{\Omega}\frac{|\Delta_h u|^p}{|h|^{\alpha p}}\,\mathrm{d}x\right)^{\frac{q}{p}}\frac{\mathrm{d}h}{|h|^Q}\right)^{\frac{1}{q}} < \infty, & 1 \le q < \infty, \\ \sup\limits_{h\in\Omega}\left(\int_{\Omega}\frac{|\Delta_h u|^p}{|h|^{\alpha p}}\,\mathrm{d}x\right)^{\frac{1}{p}} < \infty, & q = \infty. \end{cases}
\end{aligned}$$

In this article, we shall write $\Delta_h u = u(x+h) - u(x)$ in short.

2.4 Basic Tools

For every $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that for all $s, t \ge 0$, there holds

$$st \le \varepsilon\, s^p + C(\varepsilon)\, t^{p'}, \tag{2.1}$$

which is the classical Young inequality. Here $\frac{1}{p} + \frac{1}{p'} = 1$. In particular,

$$ab \le \varepsilon\, a^2 + C(\varepsilon)\, b^2. \tag{2.2}$$

Let $B_R \Subset \mathbb{H}^n$ be a CC-ball, and f an integrable function on B_R, we define the average of f over the CC-ball B_R as

$$(f)_{B_R} = \fint_{B_R} f(x)\,\mathrm{d}x = \frac{1}{|B_R|}\int_{B_R} f(x)\,\mathrm{d}x \approx R^{-Q}\int_{B_R} f(x)\,\mathrm{d}x. \tag{2.3}$$

We present the definition of weak solutions. If for any $\varphi \in C_0^{\infty}(\Omega)$, there holds

$$\int_{\Omega} A(x, \mathfrak{X}u)\cdot\mathfrak{X}\varphi\,\mathrm{d}x = \int_{\Omega}|F|^{p-2}F\cdot\mathfrak{X}\varphi\,\mathrm{d}x, \tag{2.4}$$

then $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution to (1.2). Here we call φ is a test function.

2.5 Reverse Hölder type inequality

The higher integrability estimates for Laplace and p-Laplace equations are well known (see [10] and [7]). In the Heisenberg group, we have the following two results for homogeneous and non-homogeneous situations, see [14].

Lemma 2.1. *Let $u \in HW^{1,p}(\Omega)$ with $2 < p < 4$ be a weak solution to* (1.1) *under the hypotheses* (A1)-(A4). *There exists a constant $c(n, p, \nu, k, L)$, but otherwise independent of μ, of the solution u, and of the vector field $A(x, \nabla u)$, such that the following inequalities hold for any* CC*-ball $B_R \Subset \Omega$:*

$$\sup_{B_{\frac{R}{2}}}|\mathfrak{X}u| \le c\left(\fint_{B_R}(\mu + |\mathfrak{X}u|)^p\,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{2.5}$$

Lemma 2.2. *Let* $u \in HW^{1,p}(\Omega)$ *with* $2 < p < 4$ *be a weak solution to equation* (1.2). *Assume that* (A1)-(A3) *and* (A5) *hold. If* $F \in L^q_{\mathrm{loc}}(\Omega)$*, then* $\mathfrak{X}u \in L^q_{\mathrm{loc}}(\Omega)$*, where* $q \in (p, \infty)$*. Moreover, there exists a positive constant* $C(n, p, \nu, L, q, a)$ *such that*

$$\left(\fint_{B_{\frac{R}{2}}} |\mathfrak{X}u|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\fint_{B_R} (\mu + |\mathfrak{X}u|)^p \,\mathrm{d}x\right)^{\frac{1}{p}} + C\left(\fint_{B_R} |F|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \tag{2.6}$$

for any CC-*ball* $B_R \Subset \Omega$.

3 Proofs of Theorem 1.1

In this section we present the proofs of Theorem 1.1. Inspired by [5], for the vector field $A(x,\xi)$ appeared in (1.2), we introduce

$$A_B(\xi) = \fint_B A(x,\xi)\,\mathrm{d}x \tag{3.1}$$

for $\xi \in \mathbb{R}^{2n}$ and a CC-ball $B \subset \Omega$. Then we define

$$V(x,B) = \sup_{\xi \in \mathbb{R}^{2n}} \frac{|A(x,\xi) - A_B(\xi)|}{\left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}}, \tag{3.2}$$

where $B \subset \Omega$ is a CC-ball and $x \in \Omega$. It follows that if $A : \Omega \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a Carathéodory vector field such that (A1)-(A4) hold, then A is locally uniformly in VMO, that is,

$$\lim_{R\to 0} \sup_{r(B)<R} \sup_{c(B)\in K} \fint_B V(x,B)\,\mathrm{d}x = 0, \tag{3.3}$$

where $K \subset \Omega$, $c(B)$ and $r(B)$ denote the center and the radius of the CC-ball B, respectively.

In order to prove Theorem 1.1, we mention that there exists a constant $\hat{C} > 0$ such that

$$\hat{C}^{-1}\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} \le \frac{|V(\xi) - V(\eta)|^2}{|\xi - \eta|^2} \le \hat{C}\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} \tag{3.4}$$

for any ξ, $\eta \in \mathbb{R}^{2n}$ and $|\xi - \eta| \ne 0$.

We are in a position to present the proof.

Proof of Theorem 1.1. We let $B_{3R} \Subset \Omega$ and select a test function $\varphi = \Delta_{-h}(\eta^2 \Delta_h u)$ to (1.1), where $\eta \in C_0^\infty(B_{3R})$ is a cut-off function satisfying that

$$0 \le \eta(x) \le 1, \quad \eta(x) \equiv 1 \text{ for } x \in B_{\frac{R}{2}}, \quad \eta(x) \equiv 0 \text{ for } x \in B_{3R} \setminus B_R, \text{ and } |\mathfrak{X}\eta| \le \frac{C}{R}.$$

One gets that

$$\begin{aligned} G_1 &= \int_{B_{2R}} [A(x+h, \mathfrak{X}u(x+h)) - A(x+h, \mathfrak{X}u)] \cdot \eta^2 \, \Delta_h \mathfrak{X}u \, \mathrm{d}x \\ &= \int_{B_{2R}} [A(x, \mathfrak{X}u) - A(x+h, \mathfrak{X}u)] \cdot \eta^2 \, \Delta_h \mathfrak{X}u \, \mathrm{d}x \\ &\quad + \int_{B_{2R}} [A(x+h, \mathfrak{X}u) - A(x+h, \mathfrak{X}u(x+h))] \cdot 2\eta \, \mathfrak{X}\eta \, \Delta_h u \, \mathrm{d}x \\ &\quad + \int_{B_{2R}} [A(x, \mathfrak{X}u) - A(x+h, \mathfrak{X}u)] \cdot 2\eta \, \mathfrak{X}\eta \, \Delta_h u \, \mathrm{d}x \\ &= G_2 + G_3 + G_4. \end{aligned} \tag{3.5}$$

We estimate each G_i $(1 \leq i \leq 4)$ in (3.5). By (A1), it is clear that

$$G_1 \geq \nu \int_{B_{2R}} \left(\mu^2 + |\mathfrak{X}u(x+h)|^2 + |\mathfrak{X}u|^2 \right)^{\frac{p-2}{2}} |\Delta_h \mathfrak{X}u|^2 \, \eta^2 \, \mathrm{d}x. \tag{3.6}$$

For G_2, according to (A4) and (2.2), we obtain that

$$\begin{aligned} G_2 &\leq \int_{B_{2R}} |h|^\alpha \, (g(x) + g(x+h)) \left(\mu^2 + |\mathfrak{X}u|^2 \right)^{\frac{p-1}{2}} |\Delta_h \mathfrak{X}u| \, \eta^2 \, \mathrm{d}x \\ &\leq \varepsilon \int_{B_{2R}} \left(\mu^2 + |\mathfrak{X}u|^2 \right)^{\frac{p-2}{2}} |\Delta_h \mathfrak{X}u|^2 \, \eta^2 \, \mathrm{d}x \\ &\quad + C \, |h|^{2\alpha} \int_{B_{2R}} (g(x) + g(x+h))^2 \left(\mu^2 + |\mathfrak{X}u|^2 \right)^{\frac{p}{2}} \mathrm{d}x \\ &\leq \varepsilon \int_{B_{2R}} \left(\mu^2 + |\mathfrak{X}u(x+h)|^2 + |\mathfrak{X}u|^2 \right)^{\frac{p-2}{2}} |\Delta_h \mathfrak{X}u|^2 \, \eta^2 \, \mathrm{d}x \\ &\quad + C \, |h|^{2\alpha} \int_{B_{2R}} (g(x) + g(x+h))^2 \left(\mu^2 + |\mathfrak{X}u|^2 \right)^{\frac{p}{2}} \mathrm{d}x. \end{aligned} \tag{3.7}$$

where $\varepsilon > 0$ will be chosen later. By (A2) and (2.2), one deduces that

$$\begin{aligned} G_3 &\leq C \int_{B_{2R}} \left(\mu^2 + |\mathfrak{X}u|^2 + |\mathfrak{X}u(x+h)|^2 \right)^{\frac{p-2}{2}} |\Delta_h \mathfrak{X}u| \; \eta \; |\mathfrak{X}\eta| \; |\Delta_h u| \, \mathrm{d}x \\ &\leq \varepsilon \int_{B_{2R}} \left(\mu^2 + |\mathfrak{X}u|^2 + |\mathfrak{X}u(x+h)|^2 \right)^{\frac{p-2}{2}} |\Delta_h \mathfrak{X}u|^2 \; \eta^2 \, \mathrm{d}x \\ &\quad + C \int_{B_{2R}} \left(\mu^2 + |\mathfrak{X}u|^2 + |\mathfrak{X}u(x+h)|^2 \right)^{\frac{p-2}{2}} |\mathfrak{X}\eta|^2 \; |\Delta_h u|^2 \, \mathrm{d}x. \end{aligned}$$

By applying Lagrange Mean Value Theorem, we obtain

$$\begin{aligned} & C \int_{B_{2R}} \left(\mu^2 + |\mathfrak{X}u|^2 + |\mathfrak{X}u(x+h)|^2 \right)^{\frac{p-2}{2}} |\mathfrak{X}\eta|^2 \; |\Delta_h u|^2 \, \mathrm{d}x \\ \leq \; & C \, |h|^2 \int_{B_{2R+|h|}} \left(\mu^2 + 2 \, |\mathfrak{X}u|^2 \right)^{\frac{p-2}{2}} |\mathfrak{X}u|^2 \, \mathrm{d}x \\ \leq \; & C \, |h|^2 \int_{B_{2R+|h|}} (\mu + |\mathfrak{X}u|)^p \, \mathrm{d}x. \end{aligned} \tag{3.8}$$

To estimate G_4, the hypothesis (A4) and (2.2) give us that

$$\begin{aligned}
G_4 &\le C\int_{B_{2R}} |h|^{\alpha}\left(g(x)+g(x+h)\right)\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p-1}{2}}\eta\,|\mathfrak{X}\eta|\,|\Delta_h u|\,\mathrm{d}x \\
&\le \varepsilon\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}\eta^2\,|\Delta_h u|^2\,\mathrm{d}x \\
&\quad + C\,|h|^{2\alpha}\int_{B_{2R}}\left(g(x)+g(x+h)\right)^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x. \qquad (3.9)
\end{aligned}$$

Here we notice that

$$\varepsilon\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}\eta^2\,|\Delta_h u|^2\,\mathrm{d}x \le C\,|h|^2\int_{B_{2R+|h|}}(\mu+|\mathfrak{X}u|)^p\,\mathrm{d}x.$$

Combining the estimates of G_i and choosing ε small enough, we obtain that

$$\begin{aligned}
&\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u(x+h)|^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}|\Delta_h\mathfrak{X}u|^2\,\eta^2\,\mathrm{d}x \\
\le\ & C\,|h|^{2\alpha}\int_{B_{2R}}\left(g(x)+g(x+h)\right)^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x \\
&+ C\,|h|^2\int_{B_{2R+|h|}}(\mu+|\mathfrak{X}u|)^p\,\mathrm{d}x. \qquad (3.10)
\end{aligned}$$

By the definition of V and (3.4), one gets

$$|\Delta_h V|^2 \le C\left(\mu^2+|\mathfrak{X}u(x+h)|^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}|\Delta_h\mathfrak{X}u|^2. \qquad (3.11)$$

We integrate both sides of (3.11) on $B_{\frac{R}{2}}$ and apply the properties of η to get

$$\begin{aligned}
\int_{B_{\frac{R}{2}}}|\Delta_h V|^2\,\mathrm{d}x &\le C\int_{B_{\frac{R}{2}}}\left(\mu^2+|\mathfrak{X}u(x+h)|^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}|\Delta_h\mathfrak{X}u|^2\,\eta^2\,\mathrm{d}x \\
&\le C\,|h|^{2\alpha}\int_{B_{2R}}\left(g(x)+g(x+h)\right)^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x \\
&\quad + C\,|h|^2\int_{B_{2R+|h|}}(\mu+|\mathfrak{X}u|)^p\,\mathrm{d}x. \qquad (3.12)
\end{aligned}$$

Dividing both sides of (3.12) by $|h|^{2\alpha}$, it follows that

$$\begin{aligned}
\int_{B_{\frac{R}{2}}}\left|\frac{\Delta_h V}{|h|^{\alpha}}\right|^2\mathrm{d}x &\le C\int_{B_{2R}}\left(g(x)+g(x+h)\right)^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x \\
&\quad + C\,|h|^{2-2\alpha}\int_{B_{2R+|h|}}(\mu+|\mathfrak{X}u|)^p\,\mathrm{d}x \\
&=: P_1+P_2. \qquad (3.13)
\end{aligned}$$

Finally, we shall give the proof that P_i is bounded for each i. In view of Lemma 2.1, ones get $|\mathfrak{X}u|^p \in L^t(\Omega)$ with $t > 1$. In particular, $|\mathfrak{X}u|^p \in L^{\frac{Q}{Q-2\alpha}}(\Omega)$. By choosing $0 < |h| < \delta < R$ and (A4), we acquire

$$\begin{aligned} P_1 &\leq C\left(\int_{B_{2R}} (g(x)+g(x+h))^{\frac{Q}{\alpha}}\,\mathrm{d}x\right)^{\frac{2\alpha}{Q}} \left(\int_{B_{2R}} \left[\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\right]^{\frac{Q}{Q-2\alpha}}\mathrm{d}x\right)^{\frac{Q-2\alpha}{Q}} \\ &\leq C\left(\int_{B_{2R+|h|}} g(x)^{\frac{Q}{\alpha}}\,\mathrm{d}x\right)^{\frac{2\alpha}{Q}} \left(\int_{B_{2R}} \left[\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\right]^{\frac{Q}{Q-2\alpha}}\mathrm{d}x\right)^{\frac{n-2\alpha}{Q}} < \infty. \end{aligned}$$

Because $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$, we get $P_2 < \infty$. It follows that $\sup\limits_{|h|<\delta} \int_{B_{\frac{R}{2}}} \left|\frac{\Delta_h V}{|h|^\alpha}\right|^2 \mathrm{d}x < \infty$ with $\delta < R$, that is, $V(\mathfrak{X}u) \in B^\alpha_{2,\infty}(\Omega)$ locally. □

4 Proofs of Theorem 1.2

For the non-homogeneous case, we need the following lemma.

Lemma 4.1. *Let $A : \Omega \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a Carathéodory vector field such that* (A1)-(A3) *and* (A5) *hold. Then A is locally uniformly in VMO, that is,*

$$\lim_{R\to 0} \sup_{r(B)<R} \sup_{c(B)\in K} \fint_B V(x,B)\,\mathrm{d}x = 0, \tag{4.1}$$

where $V(x,B)$ is given in (3.2)*, $K \subset \Omega$, $c(B)$ and $r(B)$ denote the center and the radius of the* CC*-ball B, respectively.*

Proof. Given a point $x \in \Omega$, we let $A_k(x) = \{y \in \Omega : 2^{-k} \leq \mathrm{dist}_{\mathrm{CC}}(x,y) < 2^{-k+1}\}$. Ones get

$$\begin{aligned} \fint_B V(x,B)\,\mathrm{d}x &\leq \fint_B \sup_{\xi\in\mathbb{R}^{2n}} \fint_B \frac{|A(x,\xi)-A(y,\xi)|}{\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}}\,\mathrm{d}y\,\mathrm{d}x \\ &= \fint_B \sup_{\xi\in\mathbb{R}^{2n}} \frac{1}{|B|}\sum_k \int_{B\cap A_k(x)} \frac{|A(x,\xi)-A(y,\xi)|}{\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}}\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \frac{1}{|B|^2}\sum_k \int_B\int_{B\cap A_k(x)} \mathrm{dist}_{\mathrm{CC}}(x,y)^\alpha (g_k(x)+g_k(y))\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \left(\frac{1}{|B|^2}\sum_k \int_B\int_{B\cap A_k(x)} \mathrm{dist}_{\mathrm{CC}}(x,y)^{\frac{Q\alpha}{Q-\alpha}}\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{Q-\alpha}{Q}} \\ &\quad\cdot\left(\frac{1}{|B|^2}\sum_k \int_B\int_{B\cap A_k(x)} (g_k(x)+g_k(y))^{\frac{Q}{\alpha}}\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{\alpha}{Q}} \\ &\leq C(Q,\alpha)|B|^{\frac{\alpha}{Q}}\left(\frac{1}{|B|^2}\sum_k \int_B\int_{B\cap A_k(x)} (g_k(x)+g_k(y))^{\frac{Q}{\alpha}}\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{\alpha}{Q}}. \end{aligned}$$

By Hölder inequality, we acquire that

$$\begin{aligned}
&\left(\frac{1}{|B|^2}\sum_k \int_B \int_{B\cap A_k(x)} (g_k(x)+g_k(y))^{\frac{Q}{\alpha}}\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{\alpha}{Q}}\\
\leq\ & C\left(\frac{1}{|B|^2}\sum_k |B\cap A_k(x)| \int_B g_k(x)^{\frac{Q}{\alpha}}\,\mathrm{d}x\right)^{\frac{\alpha}{Q}}\\
\leq\ & \frac{C}{|B|^{\frac{2}{q}}}\left(\sum_k \|g_k\|^q_{L^{\frac{Q}{\alpha}}(B)}\right)^{\frac{1}{q}} \frac{1}{|B|^{2\left(\frac{\alpha}{Q}-\frac{1}{q}\right)}}\left(\sum_k |B\cap A_k(x)|^{\frac{\alpha q}{\alpha q-Q}}\right)^{\frac{\alpha}{Q}\cdot\frac{\alpha q-Q}{\alpha Q}}\\
\leq\ & C(Q,\alpha,q)\,|B|^{-\frac{\alpha}{Q}}\left(\sum_k \|g_k\|^q_{L^{\frac{Q}{\alpha}}(B)}\right)^{\frac{1}{q}}.
\end{aligned}$$

We choose $r>0$ small enough and observe that $x\to \|g_k\|_{l^q(L^{\frac{Q}{\alpha}}(B_r(x)))}$ is continuous on the set $\{x\in\Omega : \mathrm{dist}(x,\partial\Omega)>r\}$. Therefore, there is a point $x_r\in K$ for $r>0$ small enough such that

$$\sup_{x\in K}\|g_k\|_{l^q(L^{\frac{Q}{\alpha}}(B_r(x)))} = \|g_k\|_{l^q(L^{\frac{Q}{\alpha}}(B_r(x)))}.$$

We obtain that

$$\lim_{r\to 0}\|g_k\|_{l^q(L^{\frac{Q}{\alpha}}(B_r(x)))} = \left(\sum_k \lim_{r\to 0}\left(\int_{B_r(x_r)} g_k^{\frac{Q}{\alpha}}\right)^{\frac{q\alpha}{Q}}\right)^{\frac{1}{q}}.$$

Each of the limits on the right hand side equals to 0. Hence we complete the proof. □

With the help of preceding lemma, we have the following result.

Proof of Theorem 1.2. We assume that $B_{3R+1}\Subset\Omega$, and choose a test function $\varphi=\Delta_{-h}(\eta^2\Delta_h u)$ to (1.2), where $\eta\in C_0^\infty(\Omega)$ is a cut-off function satisfying that

$$0\leq\eta(x)\leq 1,\quad \eta(x)\equiv 1 \text{ for } x\in B_{\frac{R}{2}},\quad \eta(x)\equiv 0 \text{ for } x\in B_{3R+1}\setminus B_R,\ \text{ and } |\mathfrak{X}\eta|\leq\frac{C}{R}.$$

According to the definition of weak solution and choice of test function, we obtain

$$\begin{aligned}
G_1 &= \int_{B_{2R}}[A(x+h,\mathfrak{X}u(x+h))-A(x+h,\mathfrak{X}u)]\cdot\eta^2\,\Delta_h\mathfrak{X}u\,\mathrm{d}x\\
&= \int_{B_{2R}}[A(x,\mathfrak{X}u)-A(x+h,\mathfrak{X}u)]\cdot\eta^2\,\Delta_h\mathfrak{X}u\,\mathrm{d}x\\
&\quad+\int_{B_{2R}}[A(x+h,\mathfrak{X}u)-A(x+h,\mathfrak{X}u(x+h))]\cdot 2\eta\,\mathfrak{X}\eta\,\Delta_h u\,\mathrm{d}x\\
&\quad+\int_{B_{2R}}[A(x,\mathfrak{X}u)-A(x+h,\mathfrak{X}u)]\cdot 2\eta\,\mathfrak{X}\eta\,\Delta_h u\,\mathrm{d}x\\
&\quad+\int_{B_{2R}}\Delta_h\left[|F|^{p-2}F\right]\cdot 2\eta\,\mathfrak{X}\eta\,\Delta_h u\,\mathrm{d}x+\int_{B_{2R}}\Delta_h\left[|F|^{p-2}F\right]\cdot\eta^2\,\Delta_h\mathfrak{X}u\,\mathrm{d}x\\
&= G_2+G_3+G_4+G_5+G_6. \qquad (4.2)
\end{aligned}$$

We have estimated the terms G_1 to G_4 in the proof of Theorem 1.1. Thus it remains to estimate G_5 and G_6. We apply (2.2) to get

$$
\begin{aligned}
G_5 &\leq C \int_{B_{2R}} \left| \Delta_h \left[|F|^{p-2} F \right] \right| |\Delta_h u| \; \eta \,\mathrm{d}x \\
&\leq C \; |h|^{2\alpha} \int_{B_{2R}} \left| \frac{\Delta_h \left[\, |F|^{p-2} F \, \right]}{|h|^{\alpha}} \right|^2 \mathrm{d}x + C \int_{B_{2R}} |\Delta_h u|^2 \; \eta^2 \,\mathrm{d}x.
\end{aligned}
$$

By applying the Lagrange Mean Value Theorem, the second term can be controlled by

$$
\begin{aligned}
C(\mu) \int_{B_{2R}} |\Delta_h u|^2 \; \eta^2 \,\mathrm{d}x &\leq C \int_{B_{2R}} \frac{\mu^p}{\mu^2} \; |\Delta_h u|^2 \; \eta^2 \,\mathrm{d}x \\
&\leq C \; |h|^2 \int_{B_{2R+|h|}} \frac{\left[\left(\mu^2 + |\mathfrak{X} u|^2 \right)^{\frac{1}{2}} \right]^p}{\left[\left(\mu^2 + |\mathfrak{X} u|^2 \right)^{\frac{1}{2}} \right]^2} \; |\mathfrak{X} u|^2 \,\mathrm{d}x \\
&\leq C \; |h|^2 \int_{B_{2R+|h|}} (\mu + |\mathfrak{X} u|)^p \,\mathrm{d}x.
\end{aligned}
$$

For the estimate of G_6, it is apparent that

$$
\begin{aligned}
G_6 &\leq \int_{B_{2R}} \left| \Delta_h \left[\, |F|^{p-2} F \, \right] \right| \; |\Delta_h \mathfrak{X} u| \; \eta^2 \,\mathrm{d}x \\
&\leq C \; |h|^{2\alpha} \int_{B_{2R}} \left| \frac{\Delta_h \left[\, |F|^{p-2} F \, \right]}{|h|^{\alpha}} \right|^2 \mathrm{d}x + \varepsilon \int_{B_{2R}} |\Delta_h \mathfrak{X} u|^2 \; \eta^2 \,\mathrm{d}x.
\end{aligned}
$$

Similarly, one obtains that

$$
\begin{aligned}
\varepsilon \int_{B_{2R}} |\Delta_h \mathfrak{X} u|^2 \; \eta^2 \,\mathrm{d}x &\leq \frac{\varepsilon}{\mu^{p-2}} \int_{B_{2R}} \mu^{p-2} \, |\Delta_h \mathfrak{X} u|^2 \; \eta^2 \,\mathrm{d}x \\
&\leq \frac{\varepsilon}{\mu^{p-2}} \int_{B_{2R}} \left(\mu^2 + |\mathfrak{X} u(x+h)|^2 + |\mathfrak{X} u|^2 \right)^{\frac{p-2}{2}} |\Delta_h \mathfrak{X} u|^2 \, \eta^2 \,\mathrm{d}x.
\end{aligned}
$$

Combining the estimates of G_i, we evidently have

$$
\begin{aligned}
&\left(\nu - 2\varepsilon - \frac{\varepsilon}{\mu^{p-2}} \right) \int_{B_{2R}} \left(\mu^2 + |\mathfrak{X} u(x+h)|^2 + |\mathfrak{X} u|^2 \right)^{\frac{p-2}{2}} |\Delta_h \mathfrak{X} u|^2 \, \eta^2 \,\mathrm{d}x \\
\leq \; & C \; |h|^{2\alpha} \int_{B_{2R}} (g_k(x) + g_k(x+h))^2 \left(\mu^2 + |\mathfrak{X} u|^2 \right)^{\frac{p}{2}} \mathrm{d}x \\
&+ C \; |h|^2 \int_{B_{2R+|h|}} (\mu + |\mathfrak{X} u|)^p \,\mathrm{d}x + C \; |h|^{2\alpha} \int_{B_{2R}} \left| \frac{\Delta_h \left[\, |F|^{p-2} F \, \right]}{|h|^{\alpha}} \right|^2 \mathrm{d}x. \qquad (4.3)
\end{aligned}
$$

By choosing $\varepsilon = \frac{\nu}{4+\frac{2}{\mu^{p-2}}}$, we obtain that

$$
\begin{aligned}
&\int_{B_{2R}} \left(\mu^2 + |\mathfrak{X}u(x+h)|^2 + |\mathfrak{X}u|^2\right)^{\frac{p-2}{2}} |\Delta_h \mathfrak{X}u|^2 \eta^2 \, \mathrm{d}x \\
\leq \;& C\, |h|^{2\alpha} \int_{B_{2R}} (g_k(x) + g_k(x+h))^2 \left(\mu^2 + |\mathfrak{X}u|^2\right)^{\frac{p}{2}} \mathrm{d}x \\
&+ C\, |h|^2 \int_{B_{2R+|h|}} (\mu + |\mathfrak{X}u|)^p \, \mathrm{d}x + C\, |h|^{2\alpha} \int_{B_{2R}} \left| \frac{\Delta_h \left[\, |F|^{p-2} F \,\right]}{|h|^\alpha} \right|^2 \mathrm{d}x. \qquad (4.4)
\end{aligned}
$$

Using (1.3) and (3.4), we obtain that

$$
|\Delta_h V|^2 \leq C \left(\mu^2 + |\mathfrak{X}u(x+h)|^2 + |\mathfrak{X}u|^2\right)^{\frac{p-2}{2}} |\Delta_h \mathfrak{X}u|^2 .
$$

Using (4.4), it follows that

$$
\begin{aligned}
\int_{B_{\frac{R}{2}}} |\Delta_h V|^2 \, \mathrm{d}x \;\leq\; & C\, |h|^2 \int_{B_{2R+|h|}} (\mu + |\mathfrak{X}u|)^p \, \mathrm{d}x + C\, |h|^{2\alpha} \int_{B_{2R}} \left| \frac{\Delta_h \left[\, |F|^{p-2} F \,\right]}{|h|^\alpha} \right|^2 \mathrm{d}x \\
&+ C\, |h|^{2\alpha} \int_{B_{2R}} (g_k(x) + g_k(x+h))^2 \left(\mu^2 + |\mathfrak{X}u|^2\right)^{\frac{p}{2}} \mathrm{d}x. \qquad (4.5)
\end{aligned}
$$

Dividing both sides of (4.5) by $|h|^{2\alpha}$ and applying the properties of η, one derives that

$$
\begin{aligned}
\int_{B_{\frac{R}{2}}} \left| \frac{\Delta_h V}{|h|^\alpha} \right|^2 \mathrm{d}x \;\leq\; & C\, |h|^{2-2\alpha} \int_{B_{2R+|h|}} (\mu + |\mathfrak{X}u|)^p \, \mathrm{d}x + C \int_{B_{2R}} \left| \frac{\Delta_h \left[\, |F|^{p-2} F \,\right]}{|h|^\alpha} \right|^2 \mathrm{d}x \\
&+ C \int_{B_{2R}} (g_k(x) + g_k(x+h))^2 \left(\mu^2 + |\mathfrak{X}u|^2\right)^{\frac{p}{2}} \mathrm{d}x. \qquad (4.6)
\end{aligned}
$$

By taking the power of $\frac{1}{2}$, one obtains

$$
\begin{aligned}
\left(\int_{B_{\frac{R}{2}}} \left| \frac{\Delta_h V}{|h|^\alpha} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \;\leq\; & C \left[\int_{B_{2R}} (g_k(x) + g_k(x+h))^2 \left(\mu^2 + |\mathfrak{X}u|^2\right)^{\frac{p}{2}} \mathrm{d}x \right]^{\frac{1}{2}} \\
&+ C\, |h|^{1-\alpha} \left(\int_{B_{2R+|h|}} (\mu + |\mathfrak{X}u|)^p \, \mathrm{d}x \right)^{\frac{1}{2}} \\
&+ C \left(\int_{B_{2R}} \left| \frac{\Delta_h \left[\, |F|^{p-2} F \,\right]}{|h|^\alpha} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} . \qquad (4.7)
\end{aligned}
$$

Restricting to B_δ with $0 < |h| < \delta$ and taking the L^q norm with respect to the measure $\frac{\mathrm{d}h}{|h|^Q}$,

it follows that

$$
\begin{aligned}
&\left(\int_{B_\delta}\left(\int_{B_{\frac{R}{2}}}\left|\frac{\Delta_h V}{|h|^\alpha}\right|^2 \mathrm{d}x\right)^{\frac{q}{2}}\frac{\mathrm{d}h}{|h|^Q}\right)^{\frac{1}{q}}\\
\leq\ & C\left(\int_{B_\delta}\left(\int_{B_{2R}}(g_k(x)+g_k(x+h))^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{q}{2}}\frac{\mathrm{d}h}{|h|^Q}\right)^{\frac{1}{q}}\\
&+C\left(\int_{B_\delta}|h|^{(1-\alpha)q}\left(\int_{B_{2R+|h|}}(\mu+|\mathfrak{X}u|)^p\,\mathrm{d}x\right)^{\frac{q}{p}}\frac{\mathrm{d}h}{|h|^Q}\right)^{\frac{1}{q}}\\
&+C\left(\int_{B_\delta}\left(\int_{B_{2R}}\left|\frac{\Delta_h\left[\,|F|^{p-2}F\,\right]}{|h|^\alpha}\right|^2\mathrm{d}x\right)^{\frac{q}{2}}\frac{\mathrm{d}h}{|h|^Q}\right)^{\frac{1}{q}}\\
=:\ & P_1+P_2+P_3.
\end{aligned}
\tag{4.8}
$$

We shall show that each P_i $(1\leq i\leq 3)$ is bounded. Since $B^\alpha_{2,q}(\Omega)\subset L^{\frac{2Q}{Q-2\alpha}}(\Omega)$ with $1\leq q<\frac{2Q}{Q-2\alpha}$, one has $|F|^{p-2}F\in L^{\frac{2Q}{Q-2\alpha}}(\Omega)$. By Lemma 2.2, we get $|\mathfrak{X}u|^{p-2}\,\mathfrak{X}u\in L^{\frac{2Q}{Q-2\alpha}}(\Omega)$. That is, $\mathfrak{X}u\in L^{\frac{2Q(p-1)}{Q-2\alpha}}(\Omega)$. Since

$$
\frac{2Q(p-1)}{Q-2\alpha}\geq\frac{Qp}{Q-2\alpha}\,,
$$

then we get $|\mathfrak{X}u|^p\in L^{\frac{Q}{Q-2\alpha}}(\Omega)$.

To estimate P_1, we write the L^q norm in polar coordinates. There is no harm in supposing that $\delta=1$, so $h\in B_1\cap\mathbb{R}^{2n}$ is equivalent to $h=r\xi$ for $0\leq r<1$ and ξ in the unit sphere $\mathbb{S}^{2n-1}$. Let $\mathrm{d}\sigma(\xi)$ be the surface measure on $\mathbb{S}^{2n-1}$. By letting $r_k=\frac{1}{2^k}$, we estimate P_1 as

$$
\begin{aligned}
P_1 &= \int_0^1\int_0^1\int_{\mathbb{S}^{2n-1}}\left(\int_{B_{2R}}(g_k(x+r\xi)+g_k(x))^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{q}{2}}\mathrm{d}\sigma(\xi)\frac{\mathrm{d}r}{r}\,\mathrm{d}t\\
&= \int_0^1\sum_{k=0}^{\infty}\int_{r_{k+1}}^{r_k}\int_{\mathbb{S}^{2n-1}}\left(\int_{B_{2R}}(g_k(x+r\xi)+g_k(x))^2\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{q}{2}}\mathrm{d}\sigma(\xi)\frac{\mathrm{d}r}{r}\,\mathrm{d}t\\
&\leq \int_0^1\sum_{k=0}^{\infty}\int_{r_{k+1}}^{r_k}\int_{\mathbb{S}^{2n-1}}\left\|(\tau_{r\xi}\,g_k+g_k)\,\left(\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{2}}\right\|^q_{L^2(B_{2R})}\mathrm{d}\sigma(\xi)\frac{\mathrm{d}r}{r}\,\mathrm{d}t.
\end{aligned}
$$

We note that $\tau_{r\xi}\,g_k(x)=g_k(x+r\xi)$. Since $|\mathfrak{X}u|^p\in L^{\frac{Q}{Q-2\alpha}}(\Omega)$ and $g_k\in L^{\frac{Q}{\alpha}}(\Omega)$, one gets that

$$
\begin{aligned}
&\left\|(\tau_{r\xi}\,g_k+g_k)\,\left(\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{2}}\right\|_{L^2(B_{2R})}\\
\leq\ & \left(\left[\int_{B_{2R}}(\tau_{r\xi}\,g_k+g_k)^{2\cdot\frac{Q}{2\alpha}}\mathrm{d}x\right]^{\frac{2\alpha}{Q}}\left[\int_{B_{2R}}\left(\mu^2+|\mathfrak{X}u|^2\right)^{\frac{p}{2}\cdot\frac{Q}{Q-2\alpha}}\mathrm{d}x\right]^{\frac{Q-2\alpha}{Q}}\right)^{\frac{1}{2}}
\end{aligned}
$$

$$= \ \|(\tau_{r\xi}\, g_k + g_k)\|_{L^{\frac{Q}{\alpha}}(B_{2R})} \left\| \left(\mu^2 + |\mathfrak{X}u|^2\right)^{\frac{p}{2}} \right\|^{\frac{1}{2}}_{L^{\frac{Q}{Q-2\alpha}}(B_{2R})} .$$

On the other hand, there holds

$$\|(\tau_{r\xi}\, g_k + g_k)\|_{L^{\frac{Q}{\alpha}}(B_{2R})} \leq \|g_k\|_{L^{\frac{Q}{\alpha}}((B_{2R})-r_k\xi)} + \|g_k\|_{L^{\frac{Q}{\alpha}}(B_{2R})} \leq 2\, \|g_k\|_{L^{\frac{Q}{\alpha}}(\varrho B_R)}$$

for each $\xi \in \mathbb{S}^{2n-1}$ and $r_{k+1} \leq r \leq r_k$, where $\varrho = 3 + \frac{1}{R}$. Therefore one gets

$$P_1 \ \leq\ C \left\| \left(\mu^2 + |\mathfrak{X}u|^2\right)^{\frac{p}{2}} \right\|^{\frac{1}{2}}_{L^{\frac{Q}{Q-2\alpha}}(B_{2R})} \|\{g_k\}_k\|_{l^q\left(L^{\frac{Q}{\alpha}}(\varrho B_R)\right)} < \infty.$$

In the Heisenberg group, a direct calculation gives us that

$$\begin{aligned}
\int_{B_\delta \cap \mathbb{H}^n} |h|^{(1-\alpha)q-Q}\, \mathrm{d}x \ &= \ \int_{B_\delta \cap \mathbb{R}^{2n}} \left[\int_{B_\delta \cap \mathbb{R}} \left(|z|^2 + t\right)^{\frac{(1-\alpha)q-Q}{2}} \mathrm{d}t \right] \mathrm{d}z \\
&= \ C(\alpha, q, Q) \int_{B_\delta \cap \mathbb{R}^{2n}} \left(|z|^2 + \delta^2\right)^{\frac{(1-\alpha)q-(2n+2)}{2}+1} \mathrm{d}z \\
&= \ C(\alpha, q, Q)\, \omega_{2n-1} \int_0^\delta \left(\rho^2 + \delta^2\right)^{\frac{(1-\alpha)q-2n}{2}} \rho^{2n-1}\, \mathrm{d}\rho \\
&\leq \ C(\alpha, q, Q)\, \omega_{2n-1} \int_0^\delta \rho^{(1-\alpha)q-1}\, \mathrm{d}\rho \ < \ \infty\, .
\end{aligned}$$

According to the fact that $u \in HW^{1,p}(\Omega)$, we deduce that

$$\begin{aligned}
P_2 \ &\leq \ C \left(\int_{B_\delta} |h|^{(1-\alpha)q-Q}\, \mathrm{d}x \right)^{\frac{1}{q}} \left(\int_{B_{2R+|h|}} (\mu + |\mathfrak{X}u|)^p\, \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq \ C \left(\int_0^\delta \rho^{(1-\alpha)q-1}\, \mathrm{d}\rho \right)^{\frac{1}{q}} \left(\int_{B_{2R+|h|}} (\mu + |\mathfrak{X}u|)^p\, \mathrm{d}x \right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

Because $|F|^{p-2}F \in B^{\alpha}_{2,q}(\Omega)$, it follows that

$$P_3 = C \left\| \frac{\Delta_h \left[\, |F|^{p-2}F\, \right]}{|h|^\alpha} \right\|_{L^q\left(\frac{\mathrm{d}h}{|h|^Q}; L^2(B_{2R})\right)} < \infty.$$

Therefore, we complete the proof of Theorem 1.2. □

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